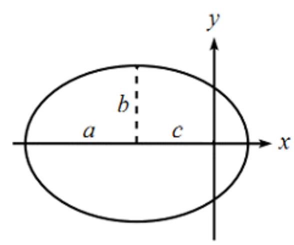
Lecture 15 - Orbit Problems

A Puzzle...

The ellipse shown below has one focus at the origin and its major axis lies along the x-axis. The ellipse has a semi-major axis of length a and a semi-minor axis of length b.
1. Write the equation of this ellipse in Cartesian coordinates
2. Find the distance c from the center to either focus of the ellipse in terms of a and b
3. The eccentricity of an ellipse is defined as $\epsilon = \frac{c}{a}$. Describe an ellipse with $\epsilon = 0$ and $\epsilon = 1$



Solution
1. The center of the ellipse lies at $(-c, 0)$, so that its equation is given by

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Essentially, an ellipse is a unit circle centered at $(-c, 0)$ whose x-dimension is stretched by a and whose y-dimension is stretched by b.

2. One property of the ellipse is that for any point (x, y) on the ellipse, the sum of distance from (x, y) to the first focus $(0, 0)$ and the distance from (x, y) to the second focus $(-2\,c, 0)$ is constant. This sum, which we call D, take the form

$$D = \sqrt{x^2 + y^2} + \sqrt{(x + 2\,c)^2 + y^2} = \text{constant} \tag{2}$$

Let us calculate D for the points $(a - c, 0)$ and $(-c, b)$:

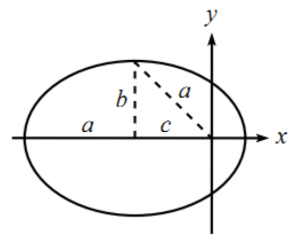
$$D = \sqrt{(a - c)^2} + \sqrt{(a + c)^2} = 2\,a \tag{3}$$

$$D = \sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} = 2\sqrt{c^2 + b^2} \tag{4}$$

The first relation tells us that the constant distance D equals to $2\,a$, which when combined with the second relation gives us the desired relation between a, b, and c

$$a^2 = b^2 + c^2 \tag{5}$$

This relation is shown in the dashed triangle within the diagram below



3. An ellipse with $\epsilon = 0$ has $c = 0$, which implies that the center of the ellipse is at the origin. Furthermore, Equation (5) implies that $a^2 = b^2$, so that the ellipse is a circle in this limit.

An ellipse with $\epsilon = 1$ has $c = a$, in which case Equation (5) implies that $a^2 = b^2 + c^2 = b^2 + a^2$ so that $b = 0$. In other words, an ellipse in this limit becomes a line segment from $(-2\,a, 0)$ to $(0, 0)$. □

Orbits

Shape of an Orbit

Recall from last time that a mass m orbiting a mass M fixed at the origin will behave as

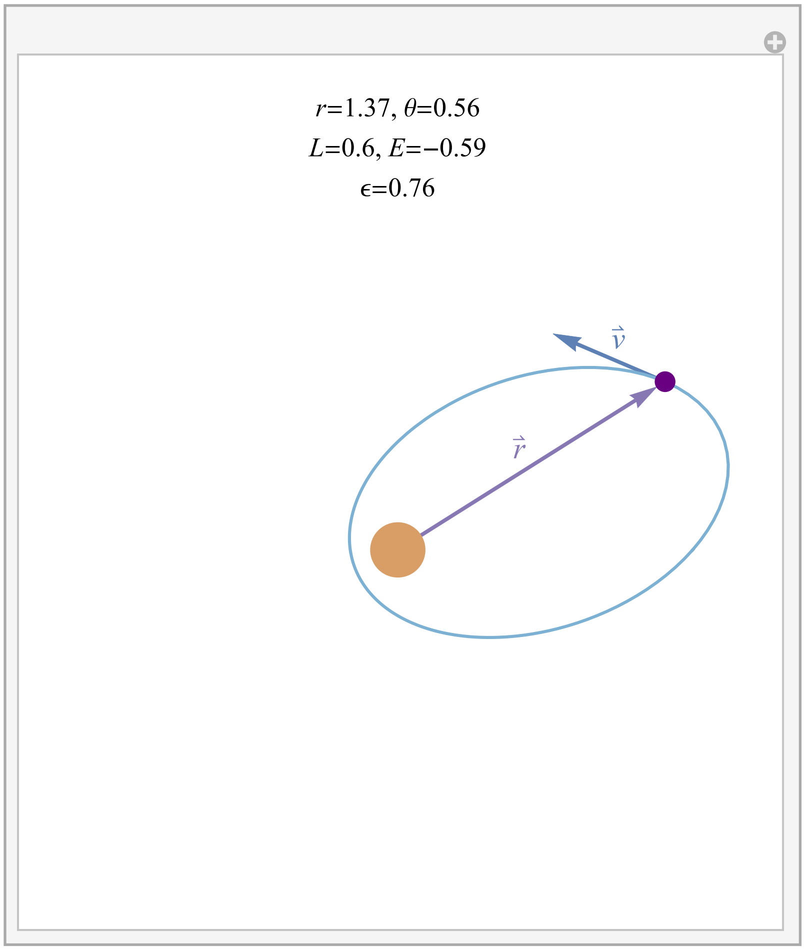
$$r[\theta] = \frac{L^2}{G\,M\,m^2}\,\frac{1}{1+\epsilon\,\mathrm{Cos}[\theta]} \tag{6}$$

where ϵ is the eccentricity

$$\epsilon \equiv \left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{7}$$

(We have chosen our axes so that the point of closest approach occurs in the $+\hat{x}$ direction.) The two variables L and E are fixed by initial conditions, and together with an initial location r and θ at time $t = 0$, these 4 variables fully specify an orbit.

The following lecture will be dedicated solely towards understanding what these orbits look like. For now, we begin by visualizing the orbits.



Let's look at some properties of this orbit. The point of closest approach occurs at $\theta = 0$ and has a distance

$$r_{\min} = \frac{L^2}{G M m^2} \frac{1}{1+\epsilon} \tag{8}$$

The furthest point from the origin reached during an orbit occurs is

$$\begin{aligned} r_{\max} &= \frac{L^2}{G M m^2} \frac{1}{1-\epsilon} \quad (\epsilon < 1) \\ r_{\max} &= \infty \qquad\qquad\quad (\epsilon \geq 1) \end{aligned} \tag{9}$$

where we have separated out the cases of closed orbits with $\epsilon < 1$ (circular or elliptical orbits) from open orbits with $\epsilon \geq 1$ (parabolic or hyperbolic orbits). When $\epsilon < 1$, $r_{\max}$ is achieved at $\theta = \pi$.

A Note about the Eccentricity

Proof of Conic Orbits

Let us prove that the orbits described by

$$r = \frac{L^2}{G M m^2} \frac{1}{1+\epsilon \operatorname{Cos}[\theta]} \tag{11}$$

are conic sections. It will be easiest to work in good old Cartesian coordinates. Defining

$$k \equiv \frac{L^2}{G M m^2} \tag{12}$$

enables us to rewrite the equation as

$$r + \epsilon x = k \tag{13}$$

where used the Cartesian x-coordinate

$$x = r\,\mathrm{Cos}[\theta] \tag{14}$$

The above equation yields $r = k - \epsilon\, x$ which we can square and substitute in $r^2 = x^2 + y^2$ to get

$$x^2 + y^2 = k^2 - 2\,k\,\epsilon\,x + \epsilon^2\,x^2 \tag{15}$$

Let's consider various cases for the eccentricity ϵ:

<u>Case 1</u> $\epsilon = 0$

The above equation reduces to the equation of a circle

$$x^2 + y^2 = k^2 \tag{16}$$

whose center is the origin (where the mass M is located).

<u>Case 2</u> $0 < \epsilon < 1$

Completing the square for the x terms and using some algebra, we can re-write the above equation as

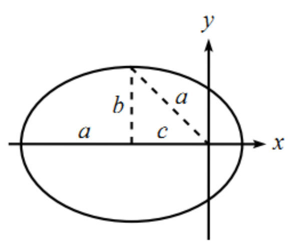
$$\frac{\left(x + \frac{k\,\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{17}$$

where

$$a = \frac{k}{1-\epsilon^2} \tag{18}$$

$$b = \frac{k}{\left(1-\epsilon^2\right)^{1/2}} \tag{19}$$

This is the equation of an ellipse with center $\left(-\frac{k\,\epsilon}{1-\epsilon^2}, 0\right)$. The semi-major and semi-minor axes are a and b, respectively, and the focal length equals $c = \sqrt{a^2 - b^2} = \frac{k\,\epsilon}{1-\epsilon^2}$. Therefore, one focus is located at the origin (where the mass M is located). The eccentricity equals $\epsilon = \frac{c}{a}$.

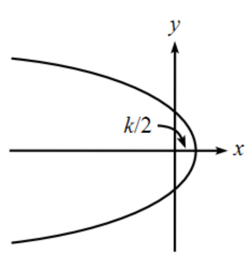


<u>Case 3</u> $\epsilon = 1$

The above equation reduces to a parabola

$$y^2 = k^2 - 2\,k\,x = -2\,k\left(x - \frac{k}{2}\right) \tag{20}$$

This is the equation of a parabola with vertex $\left(\frac{k}{2}, 0\right)$ and focal length $\frac{k}{2}$ (the focal length of a parabola written as $y^2 = 4\,a\,x$ is a). Therefore, the orbit is a parabola with a focus at the origin (where mass M is located).



<u>Case 4</u> $1 < \epsilon$

Completing the square for the x terms, our equation becomes

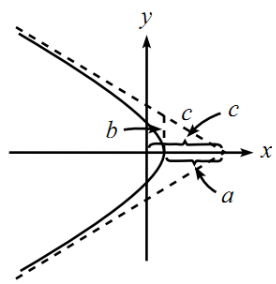
$$\frac{\left(x-\frac{k\epsilon}{\epsilon^2-1}\right)^2}{a^2}-\frac{y^2}{b^2}=1 \tag{21}$$

where

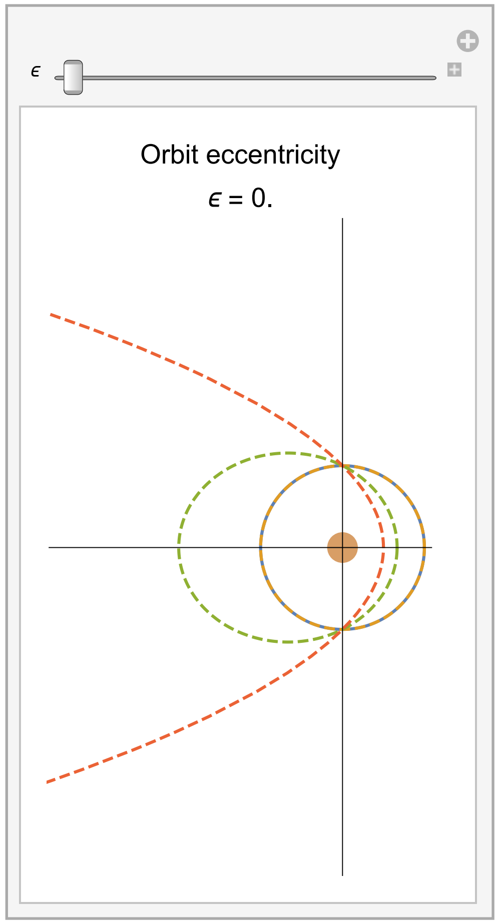
$$a=\frac{k}{\epsilon^2-1} \tag{22}$$

$$b=\frac{k}{\left(\epsilon^2-1\right)^{1/2}} \tag{23}$$

This is the equation for a hyperbola with its center (defined as the intersection of its asymptotes) located at $\left(\frac{k\epsilon}{\epsilon^2-1},\, 0\right)$. The focal length equals $c=\sqrt{a^2+b^2}=\frac{k\epsilon}{\epsilon^2-1}$. Therefore, the hyperbola has a focus at the origin. Note that the eccentricity equals $\epsilon=\frac{c}{a}$.



The following Manipulate puts it all together, demonstrating how an orbit (blue) around a sun fixed at the origin changes from a circle (dashed gold, $\epsilon=0$) to an ellipse (dashed green, $\epsilon=0.5$) to a parabola (dashed red, $\epsilon=1$) to a hyperbola ($\epsilon>1$).



Kepler’s Laws

From the above results, it is straightforward to derive Kepler’s three laws.

Kepler’s First Law

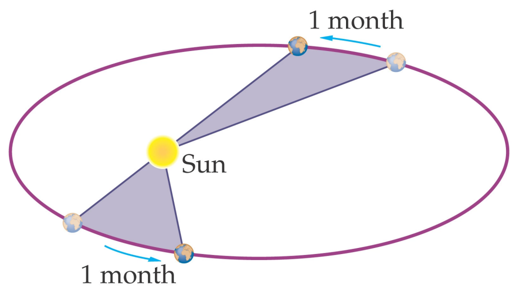
The planets move in elliptical orbits with the sun at one focus.

Proof

We proved this in the section "Proof of Conic Orbits" when the eccentricity $0 < \epsilon < 1$. The special case $\epsilon = 0$ of a circle also obeys this fact (the two foci of a circle are both at its center). Although there are flying objects moving past the sun in hyperbolic (or perhaps even parabolic) orbits, we don’t call them planets because we only see them once. □

Kepler’s Second Law

The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit.

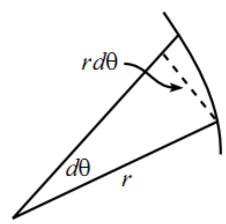


Proof

This law is nothing other than the statement of conservation of angular momentum. The area swept out during a short period of time equals

$$dA = r\,(r\,d\theta)\,/\,2 \tag{24}$$

since we the area is (to first order) a thin triangle with legs r and $r\,d\theta$.



Using $L = m\,r^2\,\dot{\theta}$,

$$\frac{dA}{dt} = \frac{1}{2}\,r^2\,\dot{\theta} = \frac{L}{2\,m} \tag{25}$$

which is constant. □

Kepler’s Third Law

The square of the period of an orbit, T, is proportional to the cube of the semi-major axis length, a. More precisely

$$T^2 = \frac{4\,\pi^2\,a^3}{G\,M} \tag{26}$$

where M is the mass of the sun.

Proof

From our proof of the second law, we found $\frac{dA}{dt} = \frac{L}{2\,m}$. Integrating over an entire orbit,

$$A = \frac{L\,T}{2\,m} \tag{27}$$

Using the first law, the area A is that of an ellipse. Denoting its semi-major and semi-minor axes by a and $b = a\,\sqrt{1-\epsilon^2}$, $A = \pi\,a\,b = \pi\,a^2\,\sqrt{1-\epsilon^2}$. Substituting in and rearranging terms,

$$\pi^2\,a^4\,(1-\epsilon^2) = \frac{L^2\,T^2}{4\,m^2} \tag{28}$$

$$T^2 = \frac{4\,\pi^2\,m^2}{L^2}\,a^4\,(1-\epsilon^2) \tag{29}$$

From the section "Proof of Conic Orbits" above, we denoted $k = \frac{L^2}{G\,M\,m^2}$ and found that the orbit is an ellipse with semi-major axis $a = \frac{k}{1-\epsilon^2} = \frac{L^2}{G\,M\,m^2}\,\frac{1}{1-\epsilon^2}$. Substituting into the above equation to eliminate $1-\epsilon^2$,

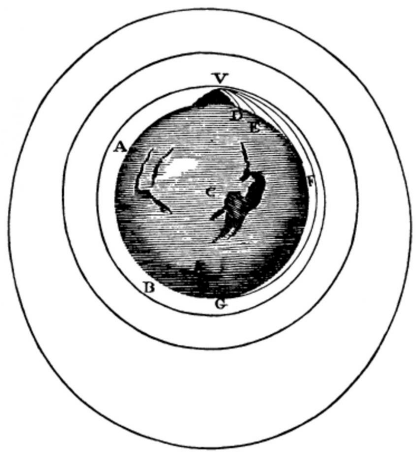
$$T^2 = \frac{4\,\pi^2\,a^3}{G\,M} \tag{30}$$

It is amazing that the mass m of the planet does not appear in this equation! □

Remembering Kepler’s Third Law

Newton’s Diagram

Generations of physics students have studied this brilliant thought experiment by Newton: what happens if you stood up on a mountain at the top of the Earth and threw a stone horizontally with a velocity v? As you increase v, you expect that the stone will fly further and further until at one critical velocity you manage to throw the stone hard enough so that it orbits the Earth in circular motion.

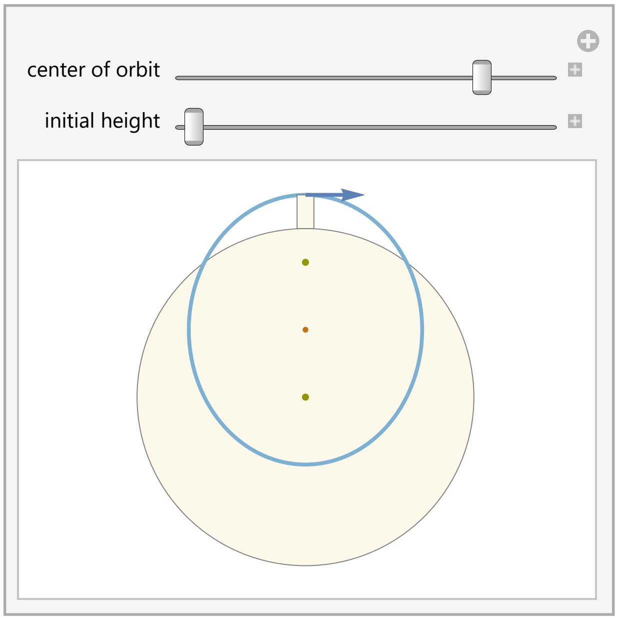


Unfortunately, generations of physics students have also gotten the wrong impression that by varying v, you get the stone to land at any point on the Earth. In particular, students imagine that for some v, you can throw a stone so that it travels $\frac{3}{4}$ of the way around the Earth before landing. Unfortunately, this is impossible!

To understand why, we will need to call upon Kepler’s First Law (which will be proved in Chapter 16): *the stone travels in an ellipse with the Earth’s center as one of its foci*.

Let us orient our axes so that $+\hat{x}$ points to the right and $+\hat{y}$ points up away from the center of the Earth. Given Kepler’s first law, the rock must travel in an ellipse. Because the initial velocity is directly horizontal - perpendicular to the line between yourself and the center of the Earth - this implies that:
1. The starting point of the rock will be the highest point
2. The semi-major and semi-minor axes of the ellipse must be along the $\hat{y}$ and $\hat{x}$ directions, respectively
3. The orbit of the particle must be symmetric in the x-direction

Here we see what the orbits can look like, with the center of the ellipse shown in orange and the two foci shown in dark green:



If we look closely at Newton's drawing, we see that it does look correct. However, note that:
1. If the rock travels $\frac{1}{4}$ of the way around the world, its final velocity as it crashes to the ground will not be parallel to the ground but instead will point towards the Earth
2. If the rock travels more than $\frac{1}{2}$ way around the world, then it must be in an orbit and will not reach land until it returns to its starting position. In particular, a rock cannot be thrown $\frac{3}{4}$ of the way around the world!

Escape Velocity

One of the reasons why it is very convenient to set $r = \infty$ as the zero point of our gravitational potential $V_{\text{grav}} = -\frac{G M m}{r}$ is because it gives us a very simple way to define escape velocity.

Example
(1) A rocket is shot radially outwards away from Earth. What would its initial velocity need to be for it to escape out to $r = \infty$?
(2) What is the escape velocity if the rocket is shot tangentially from the surface of the Earth?

Solution
(1) The energy of the rocket on the Earth's surface would be

$$E = \frac{1}{2} m v_{\text{escape}}^2 - \frac{G M_E m}{R_E} \tag{36}$$

The minimum energy required for this rocket to escape out to infinity would require for it to just barely reach $r = \infty$ with no remaining kinetic energy (i.e. it must turn all of its kinetic energy into translational energy to reach $r = \infty$). Therefore we must have

$$E = 0 - \frac{G M_E m}{\infty} = 0 \tag{37}$$

So the defining condition for escape velocity is $E = 0$, which gives us a simple way to solve for escape velocity from Earth's surface

$$0 = \frac{1}{2} m v_{\text{escape}}^2 - \frac{G M_E m}{R_E} \tag{38}$$

Using $M_E = 6 \times 10^{24}$ kg and $R_E = 6.4 \times 10^6\ m$, we find $v_{\text{escape}} = \left(\frac{2 G M_E}{R_E}\right)^{1/2} = 11\,100\ \frac{m}{s}$.

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M = 6 × 10^24; R = 6.4 × 10^6; G = 6.67 * 10^-11; m = 1;
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$$\sqrt{\frac{2\,G\,M}{R}}$$

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(2) More generally, if the rocket was shot in any direction relative to the Earth's surface, its orbit motion would follow the orbit equation we derived last time,

$$\frac{1}{r} = \frac{G\,M\,m^2}{L^2}\,(1+\epsilon\,\mathrm{Cos}[\theta]) \tag{39}$$

$$\epsilon = \left(1+\frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{40}$$

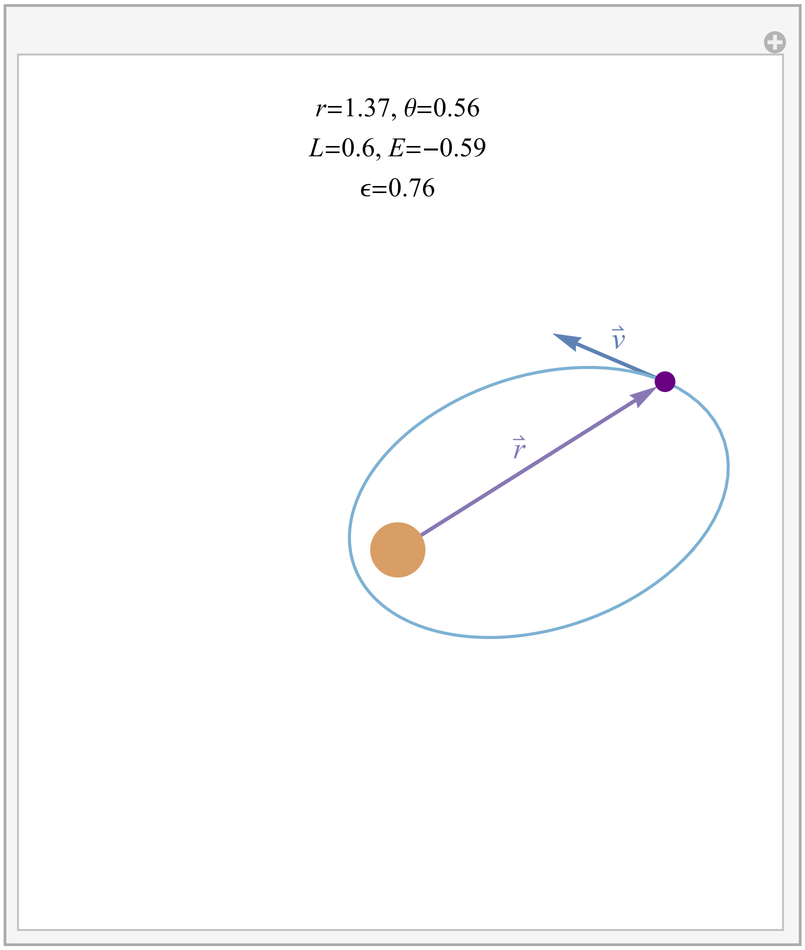
An orbit with $\epsilon \geq 1$ would escape out to infinity, which requires $E \geq 0$. Thus, the minimum energy needed to escape is $E = 0$, which implies the same escape velocity $v_{\text{escape}} = \left(\frac{2\,G\,M_E}{R_E}\right)^{1/2}$ found above. In other words, having a speed v_{escape} pointing in any direction at the surface of the Earth allows you to escape to $r = \infty$ (provided that your orbit does not intersect the Earth, in which case you will crash and burn). □

Play with Orbits

The following Manipulate allows you to play around with the initial conditions $\vec{r}$ and $\vec{v}$ and see what the corresponding orbit looks like. You can drag the position and velocity vectors $\vec{r}$ and $\vec{v}$. I allow all orbits (in our derivation, we assumed $\theta_0 = 0$ so that the point of closest approach occurs on the $+x$-axis). I have set the constants $G = M = m = 1$.

In particular, consider these questions:

1. Suppose you are at an arbitrary point in the orbit with an $\vec{r}$ and $\vec{v}$, and now you instantaneously change $\vec{v} \to \vec{v}_{\text{new}}$. What point do you know that the new orbit passes through?
2. When you are at the closest point to M in an orbit and you change $\vec{v}$ by some scalar factor $\vec{v} \to \alpha\,\vec{v}$, how does the orbit change?
3. Suppose you are orbiting a spaceship around M in a large elliptical orbit. One day, you desired to land on the planet and sight-see for a bit. In what direction should you fire your thrusters to allow your spaceship to reach the planet the fastest?
4. Suppose a mass is undergoing circular orbital motion around the Earth at a radius r_1, and you now want to transition to a circular orbit at a radius r_2 (where $r_2 > r_1$). How could you do it?



Let’s discuss the answer to the above questions. However, the best way to learn is to play with the interface above. In the following, $\vec{r}$ and $\vec{v}$ refer to the *initial* position and velocity, respectively.

1. The only point that we are guaranteed to pass through is the initial point $\vec{r}$. If the new orbit is a closed orbit (a circle or ellipse), the new orbit will pass through $\vec{r}$ infinitely many times; if the new orbit is open the particle will start at $\vec{r}$ and go out to infinity.

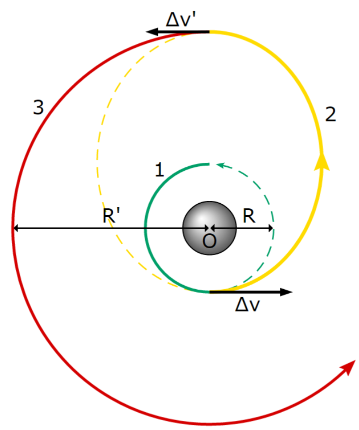
2. Assume that we have an orbit with $L \neq 0$ so that $\vec{r}$ and $\vec{v}$ are not parallel. If we increase $\vec{v}$ so that $|\vec{v}| = \infty$, we will have a hyperbolic orbit (since the energy will be $E > 0$). If we then reduce $|\vec{v}|$, we will transition to a parabolic orbit, then an elliptical orbit (let’s call this "ellipse type 1"), then a circular orbit, then an elliptical orbit (let’s call this “ellipse type 2”), and finally when $\vec{v} = \vec{0}$ we obtain a straight line plowing directly into the planet (which has $\epsilon = 1$, so we will call this a "parabolic" orbit).

Ellipse type 1 has $r_{\text{min}} = |\vec{r}|$. Thus, provided that $\vec{r}$ is above the planet, the orbit will never crash into the planet. One of the foci of the elliptical orbit is at the origin - i.e. the sun - by Kepler’s 1st law; the other foci is on the far side of the sun. As $|\vec{v}|$ is decreased, the other foci approaches the origin until the two coincide and you get circular orbit. If you continue decreasing $|\vec{v}|$, you reach an ellipse type 2.

Ellipse type 1 has $r_{\text{max}} = |\vec{r}|$, so an orbit that starts off above the Earth may still crash into it. (This is the orbit that any object you throw will follow.) As you continue to decrease $|\vec{v}|$, the focus not located at the origin will approach $\vec{r}$ and you will get a very sharp and narrow orbit.

3. As you can see by playing around with the Manipulate above, you want to fire your thrusters in the direction of $\vec{v}$ in order to decrease your velocity’s magnitude as much as possible. Doing so will decrease your angular momen-

tum L, and that will be the fastest track to reaching your planet.
Note that this is very different from our intuition on Earth where we would simply try and increase our velocity in the direction of our intended target; doing so in an orbit would end up skewing your motion and increasing your energy (doing things this way would force you to fight gravity rather than using it in your favor). I discuss this problem more fully in "Advanced Section: Changing Orbits" below.
4. The Hohmann-transfer orbit is a solution to this problem, which has actually been implemented during space flights such as the Apollo missions. First, you thrust your boosters to increase your velocity, bringing you into an elliptical orbit whose maximum radius is r_2. Then when you reach that distance you once again accelerate to bring yourself into a circular orbit at radius r_2.



Fixed M Assumption

Near-Earth Limit: Parabolic Path of Projectile Motion

Energy of Elliptical Orbits

Advanced Section: Changing Orbits

Visualizing Orbits

PHeT Colorado made an incredible orbit simulator! Use this interactive platform to explore the following questions:

- Select the *Model* tab and consider the Sun/Earth system (the default system):
    - If the sun became more massive, how would the Earth's orbit change?
    - If the Earth became more massive, how would its orbit change?
    - Pause the simulation, change the velocity of the Earth, and resume the simulation. Will the new orbit necessarily come back to the point at which you changed the Earth’s velocity?
    - Can you make the Earth escape the Sun’s gravitational field?
- Change to the Sun/Earth/Moon in the upper right
    - If the Sun were made 1.5 times as massive, what happens to the moon?

- Explore what happens if the Earth is made only slightly more massive. How much more massive can you make the Earth and still retain the moon for at least 3 orbits?
- Explore what happens if the Earth is made less massive. Be gentle, as the moon can easily be lost
- The moral of the story is that the 3-body problem is not stable! Small perturbations to the system can cause the moon to easily get lost.

Mathematica Initialization